ON THE TWO GENERAL RECIPROCAL METHODS IN GRAPHICAL STATICS.

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§ 1.

The methods employed in the geometrical or graphical solution of statical problems are, in fact, merely applications of the parallelogram of forces, so systematized and combined that the skilful draughtsman is able, by these geometrical processes alone, to make computations sufficiently exact for practical purposes with a rapidity and insight into the real relations of the quantities treated which often far surpasses that of any algebraic or numerical process.

These geometrical processes are of two principal kinds. The first kind determines the stress in each piece of any given framed structure, in which the stresses are determinate, by drawing a reticulated polygon whose lines represent these stresses and the applied forces. If a certain order of procedure be observed in drawing this reticulated force polygon, the frame and force polygon stand in a reciprocal relationship to each other, which has been clearly set forth elsewhere.* The second kind of geometrical process employed aims at somewhat more general relations than those obtained by the force polygon, and applies not only to any framed structure considered as a single elastic piece of material, but to any elastic piece, framed or not. The greater generality attained by processes of the latter kind is due to the assumption of such arbitrary forms of framing that their geometrical properties are of material assistance in determining the magnitudes sought.

Hitherto, one process only has been known which possesses this characteristic generality, which process is based upon the properties of the *equilibrium polygon.* An equilibrium polygon is often called a catenary or funicular polygon, since it is the form assumed by a perfectly flexible string under the action of the forces applied to it. Some of its geometrical properties have

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* Reciprocal Figures. James Clerk Maxwell. Philosophical Magazine, vol. 27. London, 1864.

Le figure reciproche nelle statica grafica. Luigi Cremona. Milano, 1872.

A New General Method in Graphical Statics. Henry T. Eddy. Van Nostrand's Engineering Magazine, vol. 18. New York, 1878.

EQUILIBRIUM POLYGON METHOD.

Fig. 1.

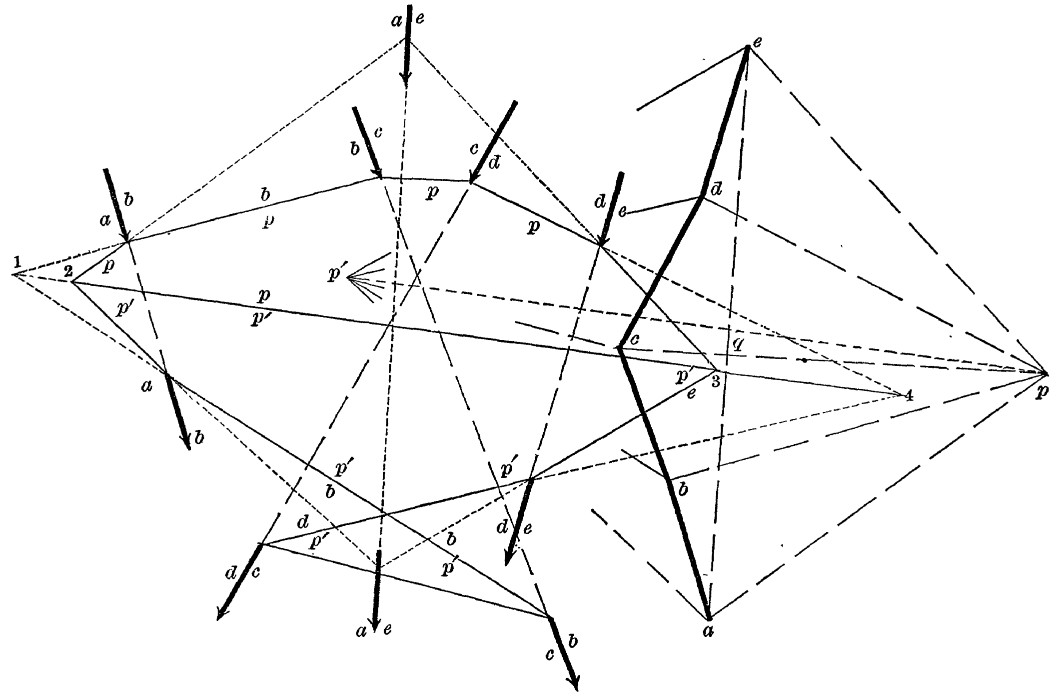
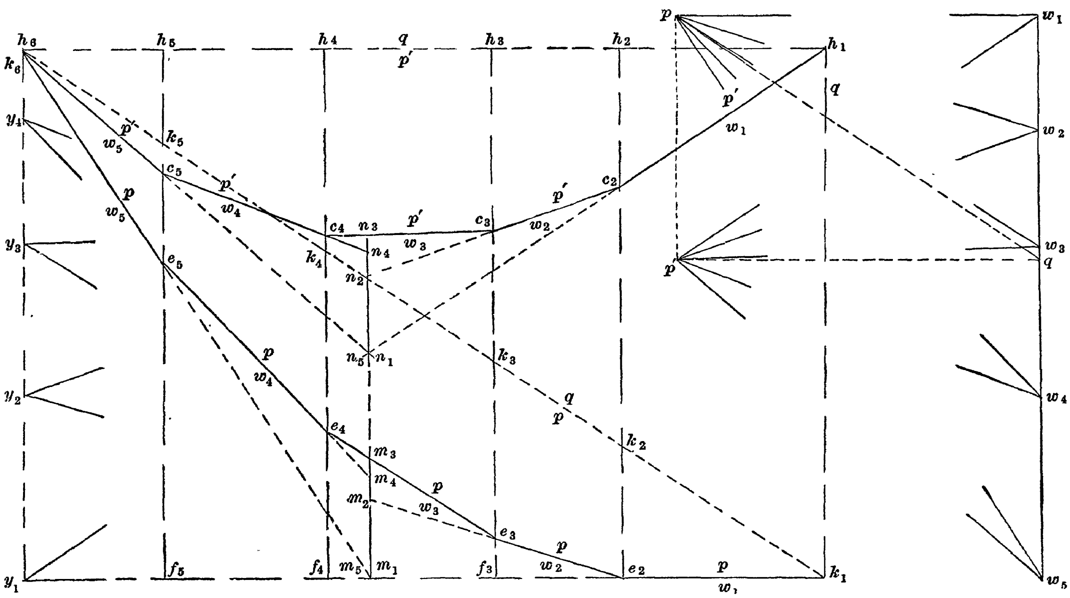


Fig. 2.



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FRAME PENCIL METHOD.

Fig. 3.

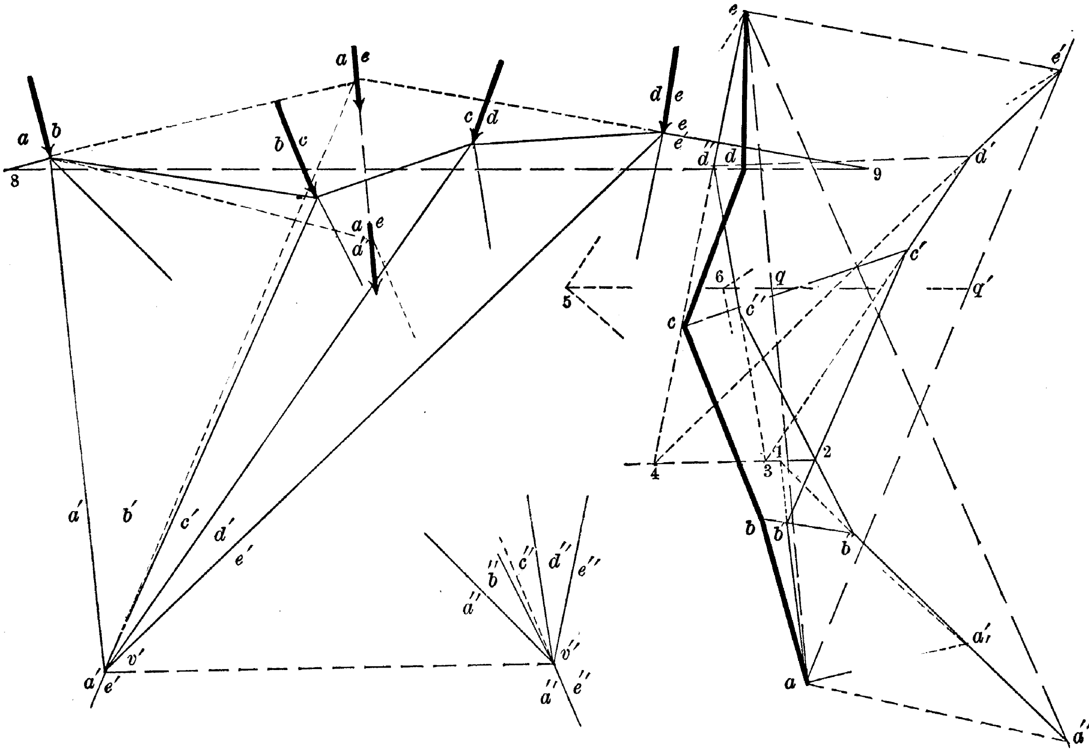
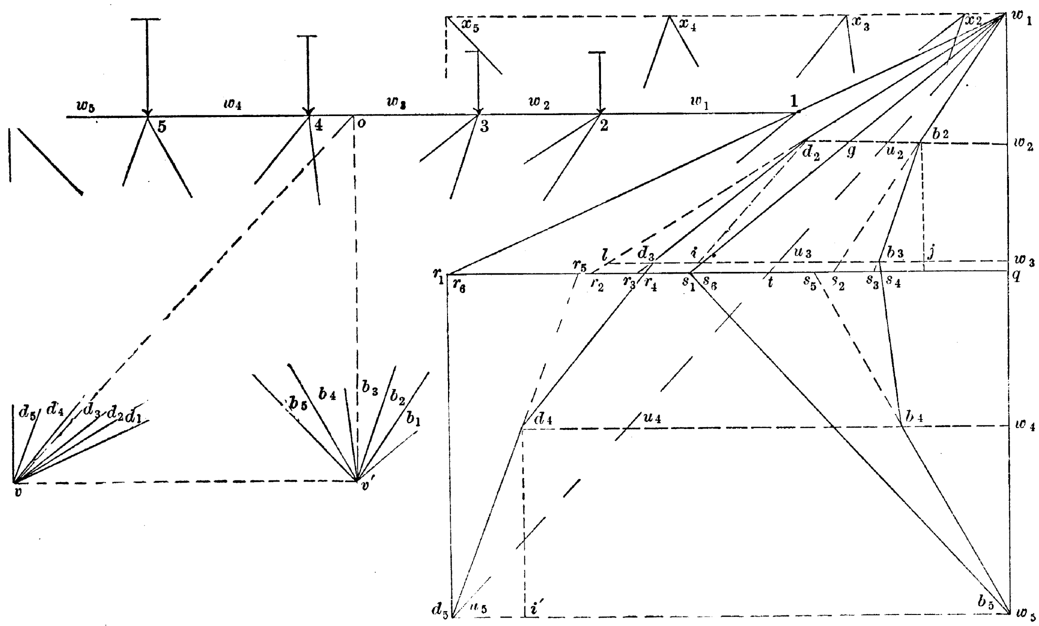


Fig. 4.



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been long known, but the systematic development and practical application of the geometrical relations involved in the equilibrium polygon and its reciprocal force polygon is comparatively recent,* and has been principally dependent upon the growth of modern higher geometry.

Culmann is justly regarded as the father of graphical statics as a practical method, and to him is due the establishment of the generality and importance of the equilibrium polygon method. Mohr discovered an important extension of the method † in showing that the deflection curve of a straight elastic girder is a second equilibrium polygon, and the author has further extended the method by showing its applicability to any curved elastic girder or arch. ‡ Many other writers have added to the subject and simplified it. The bibliography of the subject will be found in sufficient detail elsewhere. §

In a paper by Poncelet, || as given with some modifications by Woodbury, ¶ the germs of a second fundamental method appear, which is of the same general nature as that of the equilibrium polygon to which it bears a certain kind of reciprocal relationship; but neither the author nor any subsequent writer seems to have seen the possibility of establishing a general graphical method of which this special solution would be a particular case.

The purpose of this paper is to establish the general properties of this new general method, and to point out the reciprocity existing between it and that of the equilibrium polygon. In attempting this, it seems best to establish the general properties of the equilibrium polygon from mechanical considerations, instead of deriving them from higher geometry, and then to obtain the corresponding properties of the new method, which we have ventured to call the *frame pencil* method for reasons which will appear subsequently.

§ 2.

In Fig. 1, let the forces which it is proposed to treat lie in one plane and have their lines of action along the arbitrarily assumed lines, on each side of

* Graphische Statik. C. Culmann. Zurich, 1866. Also, vol. I, 2d ed. Zurich, 1875.

† Beitrag zur Theorie der Holz- und Eisenconstructionen. Zeitschr. d. Hannov. Ing. und Arch. Ver. 1868.

‡ New Constructions in Graphical Statics. Henry T. Eddy. Van Nostrand's Engineering Magazine. Vol. 16. New York, 1877.

§ Ueber die Graphische Statik. J. I. Weyrauch. Leipzig, 1874. Translated as an Introduction to The Elements of Graphical Statics. A. J. Du Bois. New York, 1875.

Lezioni di Statica Grafica. Antonio Favaro. Padova. Of this work there is a forthcoming French translation. Gauthier-Villars. Paris.

|| Mémorial de l'officier du génie. No 12.

¶ Stability of the Arch. D. P. Woodbury. New York, 1858.

which are the letters *ab*, *bc*, *cd*, *de* respectively: these lines may be called a *diagram* of the forces, and this kind of notation is employed for convenience of clearly showing the reciprocity between the diagram occupying the left of the figure and the force polygon on the right, which we shall now construct. Draw the lines at the extremities of which are *ab*, *bc*, *cd*, *de*, in such directions and of such lengths that they shall severally represent, on some convenient scale, the magnitudes as well as the directions of the forces. This will constitute the *polygon* of the applied forces. Since the force diagram shows on some convenient scale of distances the relative position and direction of the forces, and the force polygon shows their relative magnitude and also their directions, it is evident that *ab*, etc., of the diagram is parallel to *ab*, etc., of the polygon, and the number or position of the forces which can be treated is in no way restricted to those we have arbitrarily chosen to illustrate the method It is necessary, in drawing the force polygon *abcde*, to have regard to the signs of the forces, so that, in passing continuously along the polygon, the motion should be all in the direction in which the forces act, and not partly with and partly against the forces. The arrow-heads show the sense in which each force is taken, and the polygon is taken in the same sense by passing continuously from *e* to *a*.

Next, assume any pole *p* as the common point of the rays *pa*, *pb*, etc., of the force pencil *p* – *abcde*. The lengths of the rays *pa*, etc., represent, on the assumed scale of forces, the magnitudes of the stresses in the members of a frame of which we shall immediately draw a diagram; and the directions of *pa*, etc., show what directions the members of the frame must have. On the left, draw the line lying between the letters *pa* parallel to the ray *pa*:—this line, as will be seen later, is one side of an equilibrium polygon, and so it will be called the *side pa* to distinguish it from the *ray pa*. The actual position of the side is of no consequence; its parallelism to the ray is the only important consideration. From the point at which the side *pa* intersects the diagram of the force *ab*, draw the side *pb* parallel to the ray *pb*: and from the intersection of the side *pb* with the diagram of *bc* draw the side *pc* parallel to the ray *pc*. In the same manner draw a side parallel to each of the rays, so that finally the polygon *p* – *abcde* has a side parallel to each of the rays of the pencil *p* – *abcde*; and the two are said to be reciprocal figures. The reciprocity is that usually existing between a frame and its force polygon, and it is that pointed out in connection with the kind of graphical process first mentioned.

Consider the forces acting at the point *pab* of the diagram, supposing that two members *pa* and *pb* of a frame meet here in a perfectly flexible hinge joint and hold the force *ab* in equilibrium. From the parallelogram of forces it is known that the sides of the triangle *pab* represent the relative magnitudes of the forces in equilibrium at the joint *pab*. Similarly, the sides of the triangle *pbc* represent the relative magnitudes of the forces held in equilibrium at the hinge joint *pbc;* and in the same manner the forces meeting at the successive joints are represented in relative magnitude by the sides of the triangle denoted by the same letters.

The sides *pa*, *pb*, etc., together, form an equilibrium polygon, so called because such a frame requires no internal bracing in order to sustain the applied forces. In the case we have taken, it is evident that the stresses are all compressive, and the frame would form an equilibrated arch.

In the system of notation here used, *p* is regarded as denoting, on the left, the area enclosed by the equilibrium polygon, *b* as the area between the lines *ab*, *pb* and *cb*, and *c* as that between *bc*, *pc* and *dc*, whether these converge or diverge, *i. e.*, whether the space so bounded is finite or not.

Close the force polygon by drawing the side *ae*, then *ae* represents the relative magnitude of the resultant of the applied forces, or the force which will hold them in equilibrium according to the sense in which it is taken; for forces proportional to the sides of a closed polygon have no resultant. The side *ae*, also, is in the direction of the resultant, the only remaining question being as to its position. Prolong the first side *pa* until it intersects the last side *pe*, and through this intersection draw the diagram of a force *ae* parallel to the closing side *ae* of the force polygon. The diagram *ae* gives the true position of the line of application of the resultant; for, suppose the prolongations of *pa* and *pe* to be members of the frame with a hinge joint at their intersection, then the sides of the triangle *pae* represent the relative magnitudes of the forces acting at this joint, when a force is applied at this point which will hold the other applied forces in equilibrium.

Draw any line intersecting the sides *pa* and *pe* as 2 3, and also *pq* in the force polygon, parallel to 2 3; then may the points 2 and 3 be taken as fixed points of support of the equilibrium polygon or arch to which the forces are applied. This arch has the span 2 3 and the applied forces cause a thrust along 2 3, whose magnitude is given by the length of *pq*. This thrust along *pq* may be sustained by a member joining 2 3 or by these joints in virtue of

their being fixed. They, in either case, together sustain the resultant which is divided so that 2 and 3 sustain aq and qe respectively, as clearly appears from the fact that the triangles paq and qep represent the forces in equilibrium at 2 and 3 respectively. The line 2 3 is called the *closing line* of the polygon p.

Again, choose any pole p' as the common point of the force pencil $p' - abcde$. To avoid multiplying lines p' has been taken upon pq prolonged. Draw the equilibrium polygon p' whose sides are parallel to the rays of the pencil $p' - abcde$, and to still further avoid the multiplication of lines, let the side $p'a$ pass through the point 2. The sides of the polygon p' are all under tension except $p'c$, but they might all have been either in tension or all in compression had the side $p'a$ been made to pass through some point other than 2.

As shown in connection with the polygon p the first and last sides of any polygon must intersect on the diagram of the resultant, hence $p'a$ and $p'e$ intersect on the line ae already drawn.

Prolong the corresponding sides pa and $p'a$, pb and $p'b$, etc., until they intersect at 2, 1, etc., then are the points 1 2 3, etc., upon one and the same straight line. For, suppose the forces which are applied to the polygon p' to be reversed in direction, then the system applied to the polygons p and p' must together be in equilibrium; and the only bracing needed is the common member 2 3 parallel to pp', since the forces applied to p produce a thrust pq along it, and those applied to p' a thrust qp', while the reactions aq and qe at 2 and 3 are in equilibrium. A similar result holds for each of the forces separately; *e. g.* the opposed forces ab acting on p and p' may be considered to be applied at opposite points of a quadrilateral whose remaining joints are 1 and 2; the force polygon corresponding to this quadrilateral is $apbp'$; hence 1 2 is parallel to pp'. But 2 3 is parallel to pp', therefore 1 2 3 are in one and the same straight line. The same may be shown respecting the remaining intersections of corresponding sides. The intersection of pc and $p'c$ does not fall within the limits of the figure.

The properties which have been established with regard to the relations of force diagram and equilibrium polygon to the force polygon and force pencil are really of a geometric nature, and are not dependent upon the fact that they represent relationships between forces. The proposition may be stated in geometric language thus: from any points $abcd$, etc., draw lines converging to a single point as pole, also from the same number of points

1 2 3 4, etc., lying in a straight line; draw lines so that there shall be one line from each of the points 1 2 3 4, etc., parallel to each different convergent through *abcd*, etc.; then, if the common point of the convergent lines be removed parallel to the line 1 2 3 4, so that the convergents severally revolve about the points *abcd*, etc., and the lines from 1 2 3 4 which are respectively parallel to the convergents also revolve severally about 1 2 3 4, etc., the loci of any and all of the intersections of these last mentioned lines are straight lines, which are parallel to the lines joining the points *ab*, *bc*, *ac*, etc.; and conversely, if these loci are the last mentioned straight lines, they revolve about the fixed points 1 2 3 4, etc., which lie in one and the same straight line.

For convenience of reference, we have here collected the special terms by which the various parts of the reciprocal figures in the left and right are designated.

In Direction and Position.	Force Diagram,	*abcde*,	Force Polygon.	In Direction and Magnitude.
	Equilibrium Polygon,	$p-abcde$,	Force Pencil.	
	Closing Line,	$2\ 3 \parallel pq$,	Resolving Ray.	
	Diagram of Resultant,	*ae*,	Resultant Force.	

§ 3.

Most of the useful applications of graphical methods treat some system of parallel forces, in which case, the equilibrium polygon has additional properties of importance which will now be exhibited.

Let the system of parallel forces be that represented in Fig. 2, viz: let the verticals 2 3 4, etc., between $w_1 w_2 w_3$, etc., be the diagrams of the applied forces, of which the relative magnitudes are w_1w_2, w_2w_3, etc., in the force polygon on the right. The force polygon in this case becomes a straight line (often called the *weight line*) and the closing side $w_5 w_1$ of the force polygon is in the same straight line with the other sides $w_1 w_2$, etc.

Assume any pole p of the force pencil $p-ww$, and construct the equilibrium polygon $p-ww$ or *ee*, whose sides are parallel to the rays of the force pencil, in the manner which has been previously explained. Draw the closing line *kk* of the polygon *ee* through the points k_1 and h_6 where the first and last sides of the equilibrium polygon intersect the verticals 1 and 6, which last are assumed to be the lines of support for the applied forces. Draw the closing ray *pq* parallel to *kk*; then, as was before shown, it divides the resultant

force $w_1 w_5$ at q into the two parts which rest on the supports in the verticals 1 and 6. The diagram of the resultant is in the vertical mm through the intersection of the first and last sides of ee, as was also previously shown.

Choose a second pole p' from which to draw a force pencil $p'-ww$. Since this pole p' has been taken on a horizontal through q, the new closing line hh of the equilibrium polygon cc, whose sides are parallel to the rays of this pencil, will then be also horizontal. The first and last side will intersect on the vertical mm before found; and corresponding sides and diagonals of the polygons ee and cc all intersect in one and the same straight line yy, which is parallel to pp', as was previously proven. The coincidences just mentioned would, in any practical case, afford a most complete series of checks and tests of accuracy in drawing.

The line pp' and its parallel yy have, in this figure, been made vertical, so that p and p' are equidistant from ww. Designate the horizontal distance from p or p' to the weight line ww by the letter H. It happens in Fig. 2 that $pw_1 = H$, but in any case the pole distance H is the horizontal component of the force pq acting along the closing line kk.

Now by similarity of triangles,

$$k_1e_2 \;(= h_1h_2) : k_2e_2 :: pw_1 : qw_1 \quad \therefore\; H \,.\, k_2e_2 = qw_1 \,.\, h_1h_2 = M_2\,,$$

the moment of flexure, or bending moment, at the vertical 2, which would be caused in a simple straight beam or girder sustaining the given weights w_1w_2, etc., and resting upon supports in the verticals 1 and 6.

Again, from similarity of triangles,

$$k_1f_3 \;(= h_1h_3) : k_3f_3 :: H : qw_1$$
$$e_2f_3 \;(= h_2h_3) : e_3f_3 :: H : w_1w_2$$
$$\therefore\; H\,(k_3f_3 - e_3f_3) = H \,.\, k_3e_3 = qw_1 \,.\, h_1h_3 - w_1w_2 \,.\, h_2h_3 = M_3\,,$$

the moment of flexure of the simple girder at the vertical 3.

Similarly, it can be shown in general that

$$H \,.\, ke = M\,,$$

i. e. that the moment of flexure at any vertical whatever (be it one of the verticals 2 3 4, etc., or not) is the product of the assumed pole distance H and the vertical ordinate ke included between the equilibrium polygon ee and its closing line kk.

Evidently the same properties can be shown to hold respecting the vertical ordinates of the polygon cc, from which it is seen that $k_2e_2 = h_2c_2$, etc., and $H \,.\, ke = H \,.\, hc = M$.

From the foregoing it appears that the equilibrium polygon for parallel forces is a moment curve, *i. e.*, its vertical ordinate at any point of the span is proportional to the bending moment at that point in a girder sustaining the given weights, and supported by resting without constraint upon piers at its extremities.

From this demonstration it is clear that

$$H \,.\, e_3 f_3 = w_1 w_2 \,.\, h_2 h_3\,, \quad H \,.\, m_1 m_2 = w_1 w_2 \,.\, e_2 m_1\,, \quad H \,.\, y_1 y_2 = w_1 w_2 \,.\, h_2 h_6$$

are respectively the moments of the force $w_1 w_2$ about the vertical 3, about the vertical mm through the center of gravity, and about the vertical 6.

Similarly, $m_1 m_3$ is proportional to the moment of all the forces at the right, and $m_3 m_5$ to all at the left of the center of gravity, but $m_1 m_3 + m_3 m_5 = 0$, as should be the case at the center of gravity, about which the moments of the applied forces vanish.

From these considerations, it appears that the segments mm or nn of the resultant are proportional to the bending moments caused by the weights at the center of gravity of a girder sustaining the given weights and resting, without constraint, upon a single support at their center of gravity.

Also, the segments $y_1 y_2$, $y_2 y_3$, etc., are proportional to the bending moments caused at the vertical 6 by the weights $w_1 w_2$, $w_2 w_3$, etc., at the vertical 6, in a girder which sustains these weights, in case it is firmly fixed and built in at this vertical, and has no other support.

A vertical to which the resultant force is transferred, as it is in this case to yy, by aid of a couple introduced at that vertical, may be conveniently designated as a pseudo-resultant. The magnitude of the moment of the couple here introduced is $H \,.\, y_1 h_6 = y_1 m_1 \,.\, w_1 w_5$, and by it the closing line kk is often said to be moved to the position $k_1 y_1$, but it seems preferable to call this last a pseudo-closing line due to the kind of constraint and support at the vertical 6.

§ 4.

In Fig. 3, let ab, bc, cd, de be the diagrams of the system of applied forces, and $abcde$ the corresponding force polygon; choose one point arbitrarily upon the line of action of each of these forces, and join these points to any assumed vertex v' by rays of the *frame pencil* $a'b'c'd'e'$. Also, join successively the points chosen by the lines bb', cc', dd', which form sides of what may be called the *frame polygon.*

Now, consider the given forces to be sustained by the frame pencil and frame polygon as a system of bracing, which system exerts a force at the vertex v' in some direction not yet known, and also exerts a force along some arbitrarily assumed member ee' which may be regarded as forming a part of the frame polygon. From the points $bcde$ in the force polygon, draw the *force lines* bb', cc', dd', ee' parallel to the sides bb', etc., of the frame polygon, and commencing at a, draw the *equilibrating force polygon* $ab'c'd'e'$, whose sides are parallel to the successive rays of the frame pencil $a'b'c'd'e'$: then the stresses upon the rays of the frame pencil will be given in relative magnitude by the lengths of the corresponding sides of the equilibrating polygon, and the stresses upon the sides bb', etc., of the frame polygon by the lengths of the force lines bb', etc. These statements are shown to hold true from the fact that, in the right hand figure, in which lengths of lines represent magnitudes of forces, a closed polygon will be found whose sides are parallel to the directions of the forces meeting at each joint of the frame. The notation, as previously employed, will assist in the ready identification of the corresponding parts of the figures on the right and left.

If a *resultant side* ae' be drawn in the equilibrating polygon, and also parallel to it from v', a *resultant ray* $a'e'$, then the ray $a'e'$ will intersect the side ee' at a point in line of action of the resultant of the given system of forces; for this intersection is such a point that if the resultant alone be applied at it, there will be the same stresses along the members $a'e'$ and ee' as the applied forces themselves produce in those members, as appears when we consider that the triangle aee' represents the forces in equilibrium at the point in question. The diagram of the resultant is ae, parallel to the side ae of the force polygon.

If the arbitrarily assumed member ee' be revolved about its intersection with de, then the force line ee' will revolve about e, and e' will move along the side $d'e'$, and it appears that the locus of the intersection of the corresponding positions of the resultant ray $a'e'$ and the last side ee' will be the diagram of the resultant ae.

A first side aa' of the frame polygon can now be drawn corresponding to the assumed last side ee', and its significance can be readily seen as follows: draw the force line aa'_1 parallel to this side aa'; then the equilibrating polygon $ab'c'd'e'$, the first side of which passes through a, can be begun at any point of aa'_1, and the closing will be parallel to its present position (as is obvious from mechanical considerations), but the stresses in each of the members of

the frame will have been changed by this supposition. The case previously assumed was that in which the stress in the side aa' is zero.

It is usually a most convenient practical simplification to make all the sides of the frame polygon lie in one and the same straight line, which may be called the *frame line;* then, since the force lines are all parallel to it, the direction of the line aa'_1 is known at once, (it being one of these parallels), and the stresses in the rays of the frame pencil are the same, whatever be the point of aa'_1 at which the equilibrating polygon is begun.

It should be noticed that the equilibrium polygon is also one case of the frame polygon.

Suppose the points 8 and 9 of the first and last sides respectively to become fixed points of support, the thrust between these points may be sustained by a member 8 9, or by 8 and 9 regarded as abutments. To find the point q of the resultant force ae, at which it is divided into the two parts sustained at 8 and 9, draw $a6$ parallel to $v'8$, and $e'6$ parallel to $e'9$, and then through 6 draw the *resolving line* qq' parallel to 8 9.

This may be regarded as the same geometric proposition as that previously proven, in which it was shown that the locus of the intersection of the first and last sides of an equilibrium polygon (reciprocal to a given force pencil) is parallel to the resultant side of the force polygon, and is the diagram of the resultant. The proposition now is that the locus of the equilibrating polygon (reciprocal to a given frame pencil) is parallel to the closing side 8 9 of the frame polygon and is the resolving line. These two statements are geometrically equivalent.

Assume a second vertex v'', and draw the frame pencil and its corresponding equilibrating polygon $ed''c''b''a''$. The last point of the equilibrating polygon is at a'' or at a'_1 according as aa'' or aa' be taken as the arbitrary side of the frame polygon.

The intersection of the resulting ray $a''e''$, parallel to the side $a''e$, with the arbitrary side aa'' is on the diagram ae of the resultant, as has been shown. Also, if aa' be taken as the arbitrary side, it has been shown that a'_15 and $e5$, respectively parallel to $v''8$ and $v''9$, intersect upon the resolving line qq'.

Again, the corresponding sides of these two equilibrating polygons intersect at 1 2 3 4, points which are upon one and the same straight line parallel to $v'v''$; for this is the same proposition respecting two vertices and two equilibrating polygons which was previously proved respecting two poles and two equilibrium polygons.

For convenient reference, a table of the special names given to the different parts of the reciprocal figures just treated is here inserted:

In Direction and Position.		In Direction and Magnitude.
Force Diagram,	$abcde$,	Force Polygon.
Frame Pencil,	$a'b'c'd'e'$,	Equilibrating Polygon.
Frame Polygon,	bb', cc', dd', ee',	Force Lines.
Resultant Ray,	$a'e'$,	Resultant Side.
Closing Side,	$8\ 9 \parallel qq'$,	Resolving Line.
Diagram of Resultant,	ae,	Resultant Force.

§ 5.

Let the same system of parallel forces, which was treated by the equilibrium polygon method in Fig. 2, be also treated by the frame pencil method in Fig. 4. Suppose them to be applied to a horizontal girder at 2 3 4 5, and let it be supported at 1 and 6.

Use 1 6 as the frame line, and choose any vertex v, arbitrarily, from which to draw the rays of the frame pencil dd. As has been previously shown, if a resultant ray vo of the frame pencil dd be drawn from v parallel to the closing line uu of the equilibrating polygon dd, this ray intersects 1 6 at the point o, at which the diagram of the resultant intersects 1 6.

Furthermore, the lines w_1r_1 and d_5r_6, respectively parallel to the abutment rays $v1$ and $v6$ of the frame pencil, intersect on the resolving line rr, which determines the point of division q of the reactions of the supports, as was before shown.

Let the vertical distance of the vertex v from the frame line 1 6 be denoted by the letter V. It happens in Fig. 4 that $v6 = V$. When, however, the frame polygon is not straight, or, being straight, is inclined to the horizon, V has different values at the different joints of the frame polygon; but in every case V is the vertical distance of the joint under consideration above or below the vertex. This possible variation of V is found to be of practical use in certain constructions.

By similarity of triangles we have

$$1\,2 : v6 :: r_1r_2 : w_1q \quad \therefore\ V \, . \, r_1r_2 = w_1q \, . \, 1\,2 = M_2\,,$$

the bending moment of girder at 2.

Draw a line through w_1 parallel to $v3$, this line by chance coincides so nearly with w_1s_1 that we will consider that it is the line required, though it was drawn for another purpose.

Again, by similarity of triangles

$$1\,3 : v6 :: r_1s_1 : w_1q, \qquad 2\,3 : v6 :: d_2g\ (= r_3s_1) : w_1w_2$$

$$\therefore V\,(r_1s_1 - r_3s_1) = V.\,r_1r_3 = w_1q\,.\,1\,3 - w_1w_2\,.\,2\,3 = M_3$$

the bending moment at 3.

Similarly, it may be shown that

$$V.\,r_1r_n = M_n,$$

i. e., that the moment of flexure at any point of application of a force to the girder is the product of the assumed vertical distance V multiplied by the corresponding segment rr of the resolving line.

To find the moment of flexure at *any* point of the girder, draw a line tangent to the equilibrating polygon (or curve) and parallel to a ray of the frame pencil at that point, then the intercept r_1r of this tangent is such that $V.\,r_1r$ is the moment required.

Also, by similarity of triangles,

$$o2 : v6 :: u_2d_2 : w_1w_2, \qquad \therefore V.\,u_2d_2 = w_1w_2\,.\,o2$$

$$o2\ (= o3 + 3\,2) : v6 :: u_3l : w_1w_3, \qquad 3\,2 : v6 :: d_3l : w_2w_3$$

$$\therefore V\,(u_3l - d_3l) = V.\,u_3d_3 = w_1w_2\,.\,o2 + w_2w_3\,.\,o3,$$

i. e., the horizontal abscissas ud between the equilibrating polygon dd and its closing side uu multiplied by the vertical distance V, are the algebraic sum of the moments of the forces about their center of gravity. The moment of any single force about the center of gravity being the difference between two successive algebraic sums may be found thus: draw d_2i parallel to uu, then is $V.\,d_3i$ the moment of w_1w_2 about the center of gravity, as may be also proved by similarity of triangles.

Again, by proportions derived from similar triangles precisely like those already employed, it appears that

$$V.\,w_2d_2 = w_1w_2\,.\,2\,6$$

is the moment of the force w_1w_2 about the point 6. And similarly it may be shown that

$$V.\,w_3d_3 = w_1w_2\,.\,2\,6 + w_2w_3\,.\,3\,6$$

is the moment of w_1w_2 and w_2w_3 about 6.

Furthermore, as this point 6 was not specially related to the points of application 1 2 3 4, we have thus proved the following property of the equilibrating polygon: if $v6$, a pseudo-resultant ray of the frame pencil, be drawn to any point of the frame line, then the horizontal abscissas between the equilibrating polygon and a side ww parallel to that ray (which may be called a pseudo-closing side), are proportional to the sum total of the moments,

about that point, of those forces which are found between that abscissa and the end of the weight line from which this pseudo side was drawn. The difference between two successive sum totals being the moment of a single force, a parallel to the pseudo side determines at once the moment of any force about the point; *e. g.* draw d_4i' parallel to ww, therefore $V.d_5i'$ is the moment of w_4w_5 about 6.

Now move the vertex to a new position v' in the same vertical with o: this will cause the closing side of the equilibrating polygon (parallel to $v'o$) to coincide with the weight line. The new equilibrating polygon bb has its sides parallel to the rays of the frame pencil whose vertex is at v'. If V is unchanged, the abscissas and segments of the resolving line are unchanged, and vv' is horizontal. Also xx parallel to vv' contains the intersections of corresponding sides and diagonals of the equilibrating polygon. These statements are geometrically equivalent to those made and proved in connection with the equilibrium polygon and force pencil.

§ 6.

In Figs. 2 and 4 we have taken $H = V$; hence the following equalities will be found to exist:

$$k_2e_2 = r_1r_2\,,\ k_3e_3 = r_1r_3\,,\ k_4e_4 = r_1r_4\,,\ \text{etc.}$$
$$m_1m_2 = u_2d_2\,,\ m_1m_3 = u_3d_3\,,\ m_1m_4 = u_4d_4\,,\ \text{etc.}$$
$$y_1y_2 = w_2d_2\,,\ y_1y_3 = w_3d_3\,,\ y_1y_4 = w_4d_4\,,\ \text{etc.}$$
$$m_2m_3 = d_3i\,,\ \text{etc.}\quad y_4k_6 = d_5i'\,,\ \text{etc.}$$

By the use of *etc.*, we refer to the more general case of many applied forces as well as to the remaining like equalities in Figs. 2 and 4.

From these equations the nature of the relationship existing between the method of the equilibrium polygon and its force pencil and the method of the frame pencil and its equilibrating polygon becomes clear. It may be stated in words as follows:

The height of the vertex (a vertical distance) and the pole distance (a horizontal force) stand as the type of the reciprocity or correspondence to be found between the various parts of the figures. The ordinates of the equilibrium polygon (vertical distances) correspond to the segments of the resolving line (horizontal forces), each of these being proportional to the bending moments of a simple girder sustaining the given weights, and resting without constraint upon supports at its two extremities.

The segments of the resultant line (vertical distances) correspond to the abscissas of the equilibrating polygon, (horizontal forces), each of these being proportional to the bending moments of a simple girder sustaining the given weights and resting without constraint upon a support at their center of gravity.

The segments of any pseudo resultant line parallel to the resultant which are cut off by the sides of the equilibrium polygon, are proportional to the bending moments of a girder supporting the given weights and rigidly built in and supported at the point where the line intersects the girder: to these segments correspond the abscissas between the equilibrating polygon and a pseudo side of it parallel to the pseudo resultant ray.

The two different kinds of support which we have supposed, viz: support without constraint and support with constraint, can be treated in a somewhat more general manner, as appears when we consider that at any point of support there may be besides the reaction of the support a bending moment, such as would be induced, for instance, when the span in question forms part of a continuous girder, or when it is fixed at the support in a particular direction. In such a case the closing line of the equilibrium polygon is said to be moved to a new position. It seems better to call it in its new position a pseudo closing line. The ordinates between the pseudo closing line and the equilibrium polygon are proportional to the bending moments of the girder so supported. It is possible to induce such a moment at one point of support as to entirely remove the weight from the other, and cause it to exert no reaction whatever; and any intermediate case may occur in which the total weight in the span is divided between the supports in any manner whatever. When the weight is entirely supported at h_6 of Fig. 2 then y_1e_2 is the pseudo-closing line of the polygon *ee*. In that case *xx* of Fig. 4 becomes the pseudo-resolving line, and in general the ordinates between the pseudo-closing line and the equilibrium polygon correspond to the segments of the pseudo-resolving line, and are proportional to the bending moments of the girder. This general case is not represented in Figs. 2 and 4;* but the particular case shown, in which the total weight is borne by the left pier, gives the equations

$$e_3f_3 = w_1x_2\,, \quad e_4f_4 = w_1x_3\,, \quad e_5f_5 = w_1x_4\,, \text{ etc.}$$

* See page 82, Researches in Graphical Statics. Henry T. Eddy. New York, 1878.